IDEAL CAPACITORS

Capacitors are used in almost every activity of electrical engineering, yet information on capacitor characteristics is scattered through a variety of textbooks, databooks, and manufacturers literature. The following is an attempt to organize some of this information.

A typical capacitor is a two-terminal device consisting of two conductors separated by a dielectric. When a voltage difference V_o is applied to the conductors, a charge of $+Q$ will appear on one conductor and an equal and opposite charge $-Q$ on the other conductor. The capacitance C is defined as the ratio of the charge on one conductor to the potential difference.

$$C = \frac{Q}{V_o} \tag{1}$$

where C is in farads, Q is in coulombs, and V_o is in volts. Actually, one farad is a rather large capacitance, so capacitance values are usually expressed in terms of μF (10^{-6}F) or pF (10^{-12}F).

The total energy stored in a capacitor is

$$W_E = \frac{1}{2}\int_{vol} \epsilon E^2 dv = \frac{1}{2}CV_o^2 = \frac{1}{2}QV_o = \frac{1}{2}\frac{Q^2}{C} \tag{2}$$

where W_E is in joules, E is the electric field in V/m, and ϵ is the permittivity. The integral expression shows that the energy stored in a capacitor with a fixed voltage difference across it increases as the permittivity of the material increases.

The permittivity is usually expressed as the product of a relative permittivity ϵ_r and the permittivity of free space ϵ_o.

$$\epsilon = \epsilon_r \epsilon_o \tag{3}$$

where

$$\epsilon_o = 8.854 \times 10^{-12} \text{F/m} \tag{4}$$

The relative permittivity is unity for a vacuum and typically in the range of 2 to 6 for most dielectrics, as we shall discuss in more detail later.

Capacitors are frequently used in series in a circuit, as shown in Fig. 1. There will be no actual charge transfer through the dielectric material. However, the electric fields will cause a movement of charge within the series string. The battery supplies a positive charge to the left plate of C_1. This positive charge attracts an equivalent negative charge on the right plate of C_1. The movement of this negative charge leaves behind a positive charge of the same amount, which the electric field will force onto the left plate of C_2. The process continues until each capacitor has the same charge $Q = Q_s$ on its left plate. That is,

$$Q_s = Q_1 = Q_2 = Q_3 \tag{5}$$

The total voltage across the series combination is

$$V = V_1 + V_2 + V_3 \tag{6}$$

and since

$$V = \frac{Q}{C} \tag{7}$$

$$\frac{Q}{C_s} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} \tag{8}$$

which can be solved for the series capacitance C_s.

$$C_s = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}} \tag{9}$$

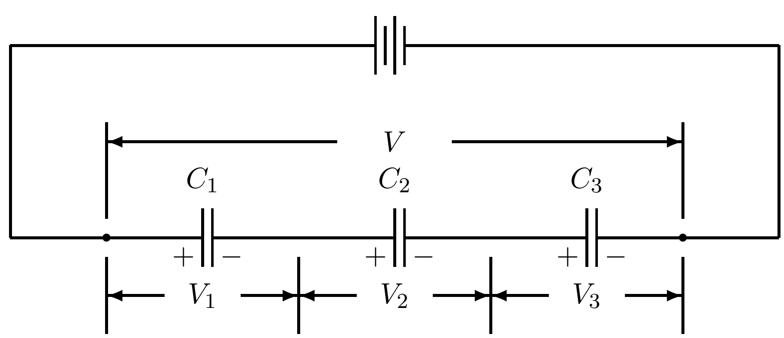

Figure 1: Capacitors in Series

A circuit of parallel capacitors is shown in Fig. 2. The voltage on each capacitor is the same and the amount of stored charge on each capacitor will be proportional to the individual capacitance values. It is not hard to show that the total parallel capacitance C_p is given by

$$C_p = C_1 + C_2 + C_3 \tag{10}$$

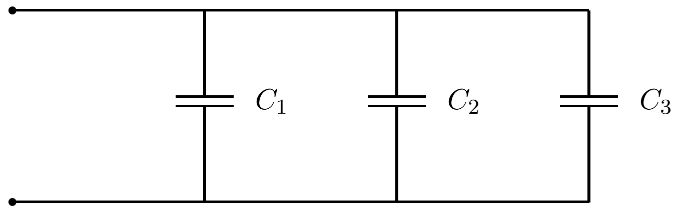

Figure 2: Capacitors in Parallel

1 Capacitance of Common Geometries

The ratio of Q to V_o depends on the geometrical arrangement of the conductors and on the electrical characteristics of the dielectric. The capacitance of a parallel plate capacitor, as illustrated in Fig. 3, is

$$C = \frac{\epsilon A}{d} \tag{11}$$

where A is the area in m^2 and d is the separation between plates. This formula is accurate only when fringing can be neglected, that is, when d is small in comparison with A.

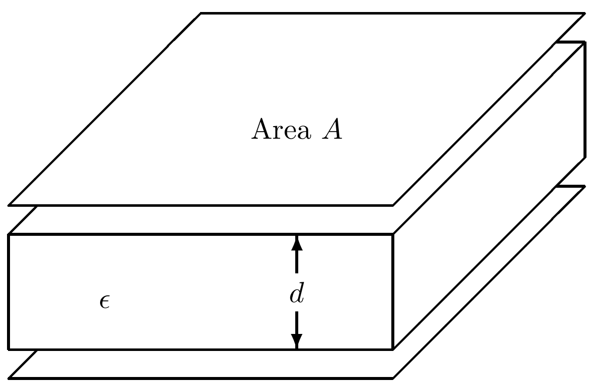

Figure 3: A Parallel Plate Capacitor

Later on, we will be interested in the capacitance of geometries where there are two different dielectrics. The simplest case is shown in Fig. 4. There is a layer of dielectric with relative permittivity $\epsilon_r > 1$ of thickness x, and a second layer of air, with thickness y. The boundary between the two dielectrics can be considered a floating electrode. In fact, we can place a conducting plate on the boundary without changing the results at all. We basically have two capacitors in series. When we solve for the series capacitance, we get

$$C = \frac{\epsilon_o A}{y} \frac{\epsilon_r}{(\epsilon_r + x/y)} \tag{12}$$

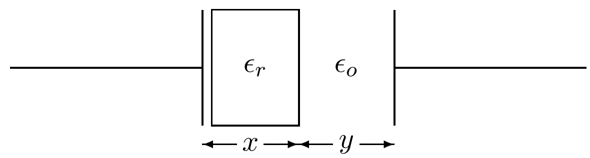

Figure 4: Capacitor with Different Dielectrics

Another important geometry is that of a coaxial cable of inner radius a, outer radius b, and length ℓ, which has capacitance

$$C = \frac{2\pi\epsilon\ell}{\ln(b/a)} \tag{13}$$

The geometry for the coaxial cable is shown in Fig. 5.

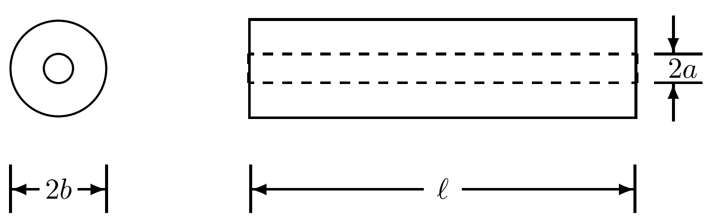

Figure 5: Coaxial Cable

The common 50 Ω coaxial cable 213/U (RG-8A/U) has a nominal capacitance of 29.5 pF/ft (96.8 pF/m). The small 75 Ω video cable 59B/U has a nominal capacitance of 21.0 pF/ft (68.9 pF/m). Most other 50 and 75 Ω cables will have capacitance values very close to these. Physically larger cables capable of carrying more power will have b and a increased in the same proportion so the ratio b/a and the capacitance will remain the same as that of a smaller cable.

Another geometry of great practical interest is the twin conductor transmission line, shown in Fig. 6a. This is composed of two conductors of radius r, with separation $2h$ between conductor centers. The conductor-to-conductor capacitance C_{cc} is given by

$$C_{cc} = \frac{\pi\epsilon\ell}{\ln[(h + \sqrt{h^2 - r^2})/r]} = \frac{\pi\epsilon\ell}{\cosh^{-1}(h/r)} \tag{14}$$

If the two conductors have a small radius and are located far apart, the expression for capacitance becomes

$$C_{cc} = \frac{\pi\epsilon\ell}{\ln(2h/r)} \tag{15}$$

The error in the approximate expression is only 5.26% when $h = 2r$ and 1.16% when $h = 3r$ so the latter equation really has a wide range of usefulness.

The conductor-to-plane capacitance C_{cp} between a cylindrical conductor of radius r and a conducting plane a distance h from the cylinder, as shown in Fig. 6b, is twice the value given by the previous two equations.

$$C_{cp} = \frac{2\pi\epsilon\ell}{\cosh^{-1}(h/r)} \approx \frac{2\pi\epsilon\ell}{\ln(2h/r)} \tag{16}$$

This equation can be used to find the capacitance between two unequal conductors. We find the capacitance of each conductor to an imaginary ground plane, and then combine the two values for C_{cp} using the formula for capacitors in series.

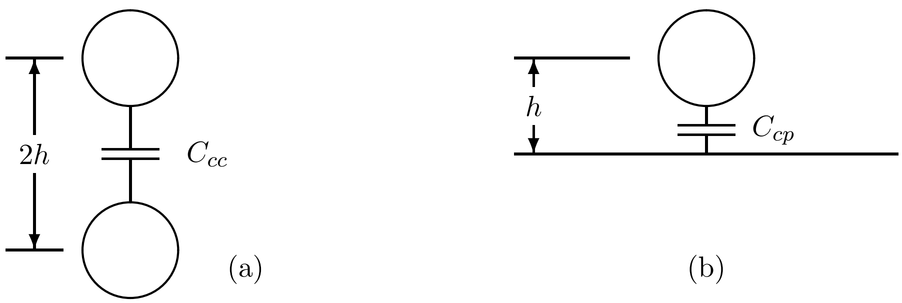

Figure 6: Twin Conductor Transmission Line

Another geometry of interest is that of a spherical capacitor of two concentric spheres with radii a and b ($b > a$) as shown in Fig. 7. It is not practical to actually build capacitors this way, but the symmetry allows an exact formula for capacitance to be calculated easily. This is done in most introductory courses of electromagnetic theory. The capacitance is given by [3, Page 165]

$$C = \frac{4\pi\epsilon}{1/a - 1/b} \tag{17}$$

If the outer sphere is made larger, the capacitance decreases, but does not go to zero. In the limit as $b \to \infty$, the *isolated* or *isotropic* capacitance of a sphere of radius a becomes

$$C_\infty = 4\pi\epsilon a \tag{18}$$

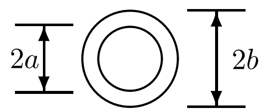

Figure 7: Spherical Capacitor

C_∞ gives us a lower bound for the capacitance of a spherical top loading element of a Tesla coil with respect to ground. One way of arriving at a reasonable estimate of the actual capacitance is to start with the isotropic capacitance C_∞ and add a correction term, as we shall see in the next section.

We will also be interested is isotropic capacitances of shapes other than spheres. Some will be difficult to impossible to calculate analytically, so we will use the isotropic capacitance of a sphere as a starting point for making an estimate. Rectangular boxes and short cylinders, for example, will have similar isotropic capacitances to a sphere. We just need to find an equivalent radius (or diameter) for these nonspherical shapes. Several different equivalents could be used, such as a geometrical mean equivalent, an arithmetic mean equivalent, or the radius of a sphere with the same surface area as the other shape. It turns out that the simplest method, the arithmetic mean, does quite well [1]. Consider a rectangular box with orthogonal edge dimensions a, b, and c. Define an equivalent sphere diameter ℓ_e where

$$\ell_e = \frac{a+b+c}{3} \tag{19}$$

For other shapes we use the dimensions of the box in which the other shape can be placed. Making the change from radius to diameter, the isotropic capacitance is now

$$C_\infty = 2\pi\epsilon\ell_e \tag{20}$$

This approach will give acceptable results in many cases. But, of course, there is no way of knowing the amount of error, or when some other approach would yield better results. It will get us in the right ballpark, however, and sometimes allow us to determine lower bounds of acceptable values obtained from other techniques.

Suppose, for example, that we wanted the capacitance between two spheres separated by several diameters. We suspect that the parallel plate capacitor formula will not be very accurate and are unable to locate a better formula. What can we do? The lower bound for the capacitance between two spheres is just half of C_∞, the isotropic capacitance for one sphere, as can be argued from Fig. 8.

The dark point at the center of the figure is obviously 'the' point at infinity that mathematicians love to talk about. If the capacitance of each sphere with respect to this 'point'

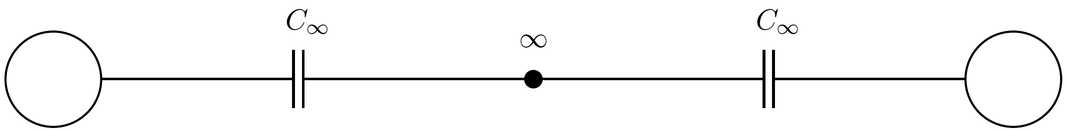

Figure 8: Capacitance Between Two Spheres

is C_∞, then the capacitance between spheres has a lower bound of $C_\infty/2$, from the formula for two capacitors in series. Bringing the spheres closer together will increase the capacitance but they cannot be separated far enough to reduce the capacitance below $C_\infty/2$.

2 Toroid Capacitance

An important emphasis of this book is the analysis of the Tesla coil. Among other things we will be interested in the capacitance of the top element (usually a 'fat' toroid) with respect to ground. We will also need the capacitance between adjacent turns (which look like 'thin' toroids) and finally the capacitance of a turn with respect to ground. As usual, we will use the published results as much as possible and leave the derivations to others.

The dimensions of a toroid are shown in Fig. 9.

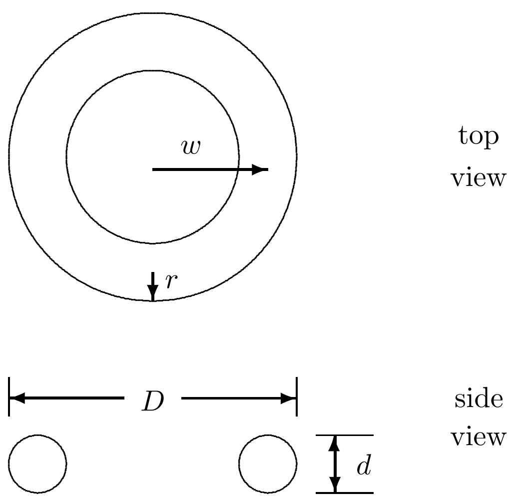

Figure 9: Toroid Dimensions

There are other coordinate systems besides rectangular, cylindrical, and spherical in which variables can be separated and Laplace's equation solved. One of these is toroidal coordinates. Moon and Spencer use this coordinate system to solve for the capacitance of an isolated toroid [2, Page 375] as

$$C_{MS} = 8\pi a\epsilon \left[\frac{Q_{-1/2}(\cosh \eta_0)}{P_{-1/2}(\cosh \eta_0)} + 2 \sum_{n=1}^{\infty} \frac{Q_{n-1/2}(\cosh \eta_0)}{P_{n-1/2}(\cosh \eta_0)} \right] \tag{21}$$

where P and Q are Legendre functions of first and second order and a and η_0 will be discussed later. The subscript 'MS' refers to Moon and Spencer, to distinguish the capacitance obtained from the value to be obtained from some empirical formulas later. The Legendre functions can be written in many different forms, as integrals or infinite series, converging for arguments greater than unity or less than unity, and so on. The non-integer subscript $(n - 1/2)$ adds another layer of complexity. Many math books do not mention the non-integer case, so one must be diligent in finding the correct expressions. The order of difficulty is much greater than for the sphere. Our mothers warned us that there would be days like this, but let us proceed.

Moon and Spencer give us an expression for $Q_{n-1/2}(\cosh \eta_0)$ [2, Page 373]

$$Q_{n-1/2}(\cosh \eta_0) = \frac{1}{\sqrt{2}} \int_0^{\pi} \frac{\cos n\theta \, d\theta}{\sqrt{(\cosh \eta_0 - \cos \theta)}} \tag{22}$$

For some reason, they do not give a similar expression for $P_{n-1/2}(\cosh \eta_0)$. However, they do give plots for both functions, which is convenient for checking computational results.

Smythe [5, Page 159] gives an expression for P in the form

$$\pi P_{n'}^{m}(x) = (n' + 1)(n' + 2) \cdots (n' + m) \int_0^{\pi} [x + \sqrt{x^2 - 1} \cos \theta]^{n'} \cos m\theta \, d\theta \tag{23}$$

This expression is valid for $x > 0$ and for any n', including 1/2, 3/2, etc. We are only interested in the case $m = 0$. If we interpret the product of terms ahead of the integral sign as a factorial with zero entries (that is with a value of unity), the expression becomes

$$P_{n'}(x) = \frac{1}{\pi} \int_0^{\pi} [x + \sqrt{x^2 - 1} \cos \theta]^{n'} \, d\theta \tag{24}$$

These expressions for P and Q can be numerically integrated using any scientific programming language (QuickBasic, etc.). An odd result is that even though the numerical integrations of Eq. 22 and Eq. 24 match the curves in Moon and Spencer, the ratio of the two numbers in Eq. 21 gives a result that is a factor of π too high. There are empirical formulas (to be given later) that make it normally unnecessary to perform all these computations, so I did not spend the time to find the error. It is most likely due to a bad calculation on my part, but I am leaning toward a typo in Moon and Spencer. At any rate, if greater accuracy is needed (or a more exalted reference), the empirical formulas should be used to make an approximate check on the computer code, with a factor of π inserted as necessary to get the predicted values within a few percent of one another.

We now return to our discussion of a and η_0. These are coordinate values in toroidal coordinates, hence need to be translated into a more familiar coordinate system. The major toroid radius w and the minor radius r are given by

$$w = a \coth \eta \tag{25}$$

$$r = \frac{a}{\sinh \eta} \tag{26}$$

Eliminating a between the two equations yields

$$w = r \sinh \eta \coth \eta = r \cosh \eta \tag{27}$$

so

$$\frac{w}{r} = \cosh \eta = x \tag{28}$$

We then solve for η as

$$\eta = \cosh^{-1} \frac{w}{r} = \ln \left(\frac{w + \sqrt{w^2 - r^2}}{r} \right) \tag{29}$$

and for a as

$$a = r \sinh \eta = \frac{r}{2}(\epsilon^{\eta} - \epsilon^{-\eta}) \tag{30}$$

This 'exact' formulation is of most interest to EM theorists and computer programmers. It will seem like overkill to most Tesla coil enthusiasts who just need to get in the right ballpark with a capacitance estimate. For this reason empirical formulas have been developed which yield an approximate value, adequate for most purposes, but obtained with much less effort. Empirical formulas for the capacitance of a toroid are given by [4]

$$C_S = \frac{1.8(D - d)}{\ln(8(D - d)/d)} \qquad (d/D < 0.25) \tag{31}$$

$$C_S = 0.37D + 0.23d \qquad (d/D > 0.25) \tag{32}$$

where D is the toroid major diameter, outside to outside, in cm, d is the toroid minor diameter in cm, and the capacitance is given in pF. Table 1 gives some isotropic capacitance values, both from the Moon and Spencer numerical integration and the empirical formulas. The deviation or error of C_S with respect to C_{MS} is given in the last column in percent.

Table 1: Isotropic Capacitance of Toroids

w	r	D	d	C_{MS}	C_S	error
meters		cm		pF	pF	%
.3	.15	90	30	40.46	40.20	−0.63
.2	.1	60	20	26.97	26.80	-0.63
.1	.05	30	10	13.49	13.40	−0.63
.2	.08	56	16	24.55	24.40	−0.62
.2	.06	52	12	22.02	21.93	−0.42
.2	.04	48	8	19.28	19.52	+1.23
.2	.02	44	4	16.00	16.43	+2.70
.2	.01	42	2	13.72	14.19	+3.42
.2	.005	41	1	12.00	12.48	+4.05
.2	.0025	40.5	.5	10.62	11.14	+4.97
.2	.001	40.2	.2	9.09	9.76	+7.36

We see that the empirical formulation agrees with Moon and Spencer to within 1% for the case of fat toroids, but gets progressively worse as the toroid gets thinner. The error is within 5% for d down to about 0.5 cm (4 gauge wire). Toroids this thin do not have the mechanical strength necessary to serve as top loading elements of a Tesla coil, so we can conclude that the empirical formulas are quite adequate for most purposes.

3 Solenoid Capacitance (Medhurst)

The isotropic capacitance of a sphere was given above as a simple formula. We looked at the theoretical formulas for capacitance of a toroid, but basically gave up and went to a simpler empirical version. After that learning experience, we will not even try to write exact equations for the isotropic capacitance of a cylinder. We will immediately write the empirical equations as developed many years ago by a man named Medhurst. These will be expressed in several different versions, to meet different needs. The simplest expression for the isotropic capacitance of a cylindrical coil of wire, with diameter D and coil length ℓ, is

$$C_M = HD \ \text{pF} \tag{33}$$

where D is in cm, and H is a multiplying factor that equals 0.51 for $\ell/D = 2$, 0.81 for $\ell/D = 5$, and varies linearly between 0.51 and 0.81 for ℓ/D between 2 and 5. Most coilers prefer values for ℓ/D between 3.5 and 4.5, so this linear range is adequate for most purposes.

An expression for H that works for ℓ/D between 2 and 8 is

$$H = 0.100976\frac{\ell}{D} + 0.30963 \quad (34)$$

Another expression for H that works for ℓ/D between 1 and 8 is

$$H = 0.0005(\frac{\ell}{D})^4 - 0.0097(\frac{\ell}{D})^3 + 0.0648(\frac{\ell}{D})^2 - 0.0757(\frac{\ell}{D}) + 0.4723 \quad (35)$$

4 Tesla Coil Capacitance

We now have expressions for the isotropic capacitance C_S of a toroid and the isotropic capacitance C_M of a coil. The next step would be to set the toroid on top the coil and add the two capacitances to get an effective capacitance C_{tc} for the Tesla coil. Unfortunately, this only works when the toroid is far away from the solenoid. As the toroid is brought near the coil form, shielding occurs such that the effective capacitance is less than the sum of the two isotropic capacitances. The Tesla coil capacitance might be written as

$$C_{tc} = C_M + KC_S \quad (36)$$

where $K < 1$. A value of $K = 0.75$ should result in a number for C_{tc} within 20% of the correct value for most Tesla coils. The resonant frequency is related to the square root of C_{tc} so a 20% error in capacitance results in only a 10% error in resonant frequency.

Most readers probably feel disappointed here. We have gone to considerable effort and still come up short of an accurate formula for C_{tc}. Our effort is not entirely wasted because we can do 'what if' analyses relatively quickly. Questions about the effect of changing coil diameter, coil length, or toroid diameter can be answered with adequate accuracy.

Someone might suggest using a modern digital capacitance meter to measure C_{tc}. This method would probably have greater error than the above formula, because the leads of the capacitance meter have a similar capacitance value as C_{tc}. Also the presence of the meter and a human body will change the capacitance.

It is possible to calculate C_{tc} numerically using Gauss's Law. If one is careful about measuring and entering all the dimensions and the locations of grounded surfaces, one should get a value for C_{tc} well within 5% of the correct value. There are programs available in the Tesla coil community that do this.

References

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