ENERGY STORAGE, EQUIVALENT REACTANCE, AND LUMPED LC CORRESPONDENCE RELATIONS FOR TESLA SECONDARY RESONATORS

PAUL NICHOLSON

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1. Introduction

This article describes the correspondence between the LC parameters used in a lumped model, and the distributed reactances of the transmission line model, with particular reference to 1/4 wave (ie series resonant) lines. We confine ourselves to loss-free 1/4 wave lines which are shorted at the base, or cold end, and open at the top, or hot end. In all the following equations, I_b is the peak base current and V_t is the peak top voltage. The constant h is used for the physical length of the line or coil, and variable x is a position variable along the line. When discussing electrical length, we will use θ and ϕ for position variables, and the line length is $\pi/2$ radians. ω is the angular frequency at resonance. For clarity, dependency of variables on position will be shown explicitly.

2. Lumped LC Model

The familiar energy balance equation

$$\frac{1}{2}LI_b^2 = \frac{1}{2}CV_t^2 \tag{2.1}$$

when combined with the formula for the self induced voltage on a lumped inductor,

$$V_t = \omega L I_b \tag{2.2}$$

leads directly to the resonant frequency formula for the lumped model,

$$\omega = \frac{1}{\sqrt{LC}} \tag{2.3}$$

In this model, the voltage along the coil is

$$V(x) = \frac{x}{h}\omega L I_b = \frac{x}{h}V_t \tag{2.4}$$

a linear rise from base to top, and the current is a uniform I_b all the along the coil.

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3. Uniform Transmission Line

In a uniform transmission line, the shunt capacitance is distributed uniformly along the length h of the line. In the uniform case with no end loading, the energy balance equation becomes

$$\frac{1}{2}\frac{L_{dc}}{\pi/2}\int_0^{\pi/2} I^2(\theta)d\theta = \frac{1}{2}\frac{C_{dc}}{\pi/2}\int_0^{\pi/2} V^2(\theta)d\theta \tag{3.1}$$

where θ is the angular position along a line of $\pi/2$ radians electrical length. L_{dc} and C_{dc} are the bulk values of series inductance and shunt capacitance as measured at low frequency. Note that we are making full use of the fact that the time variation of both the voltage and current profiles has a constant phase along the line, so the time dependancy terms can be dropped. As a result, $I(\theta)$ and $V(\phi)$ should be regarded as peak magnitude profiles, with θ measured from the shorted end.

If the uniform line is terminated in an open circuit, the current and voltage distributions both have a sinusoidal profile, thus the energy equation becomes,

$$\frac{1}{2}\frac{L_{dc}}{\pi/2}I_b^2\int_0^{\pi/2} \cos^2(\theta)d\theta = \frac{1}{2}\frac{C_{dc}}{\pi/2}V_t^2\int_0^{\pi/2} \sin^2(\theta)d\theta \tag{3.2}$$

which simplifies to

$$L_{dc}I_b^2 = C_{dc}V_t^2 \tag{3.3}$$

We now follow the same procedure as we did for the lumped case, only this time we must integrate the induced EMF along the coil to obtain the top voltage,

$$V_t = \omega\frac{L_{dc}}{\pi/2}I_b\int_0^{\pi/2} \cos(\theta)d\theta = \frac{2}{\pi}\omega L_{dc}I_b \tag{3.4}$$

Substituting this into 3.3 gives

$$\omega = \frac{\pi}{2}\frac{1}{\sqrt{L_{dc}C_{dc}}} \tag{3.5}$$

We can arrive at this formula by an alternative route: The phase constant β is given by

$$\beta = \omega\sqrt{\frac{L_{dc}C_{dc}}{h^2}} \tag{3.6}$$

where h is the length of the line, and since the line is a quarter wave, the electrical length is

$$h\beta = \frac{\pi}{2} \tag{3.7}$$

Combining these two equations to eliminate $h\beta$ also leads to the expression 3.5 for ω.

4. Non-uniform line

The above analysis fails when the line is non-uniform, since then C and L are functions of position. Thus a more general energy balance equation is necessary,

$$\frac{1}{2}\int_0^{\pi/2} L(\theta)I^2(\theta)d\theta = \frac{1}{2}\int_0^{\pi/2} C(\theta)V^2(\theta)d\theta \tag{4.1}$$

where $C(\theta)$ and $L(\theta)$ are the distributed parameters per radian length, and

$$\begin{aligned} C_{dc} &= \int_0^{\pi/2} C(\theta) d\theta \\ L_{dc} &= \int_0^{\pi/2} L(\theta) d\theta \end{aligned} \tag{4.2}$$

In this case, the voltage and current profiles are no longer sinusoidal, and we have the relations,

$$\begin{aligned} V_t &= \omega \int_0^{\pi/2} L(\theta) I(\theta) d\theta \\ I_b &= \omega \int_0^{\pi/2} C(\theta) V(\theta) d\theta \\ \frac{dV(\theta)}{d\theta} &= \omega L(\theta) I(\theta) \\ \frac{dI(\theta)}{d\theta} &= -\omega C(\theta) V(\theta) \end{aligned} \tag{4.3}$$

These equations are useful for examining the relations between V/I profiles and L/C profiles numerically. The phase constant is also now a function of position:

$$\beta(\theta) = \omega \sqrt{L(\theta) C(\theta)} \tag{4.4}$$

and the integral of the phase change $\beta(\theta) d\theta$ over the line must be $\pi/2$, so

$$\int_0^{\pi/2} \beta(\theta) d\theta = \omega \int_0^{\pi/2} \sqrt{L(\theta) C(\theta)} d\theta = \frac{\pi}{2} \tag{4.5}$$

which gives us

$$\omega = \frac{\pi/2}{\int_0^{\pi/2} \sqrt{L(\theta) C(\theta)} d\theta} \tag{4.6}$$

5. Non-uniform line with longitudinal coupling

The non-uniform analysis above is still not applicable to a tesla coil. The solenoid has non-negligible radius so it will exhibit an additional energy storage due to the internal capacitance between different parts of the coil. In addition, the solenoid mutual inductance provides yet another energy storage mechanism. Equation 4.4 is no longer valid when these longitudinal couplings are present. The internal capacitance is energised by differential voltage along the line, so that it stores a total 'internal' energy

$$\frac{1}{4} \int_0^{\pi/2} \int_0^{\pi/2} C_{int}(\theta, \phi) (V(\theta) - V(\phi))^2 d\theta d\phi \tag{5.1}$$

where $C_{int}(\theta, \phi)$ describes the internal capacitance between the infinitesimal rings of the coil at positions θ and ϕ, and $C_{int}(\theta, \phi) = C_{int}(\phi, \theta)$. The extra factor of 2 in the denominator occurs because the integral counts each contribution twice.

The total stored energy in the inductor, when mutual inductance is taken into account, is

$$\frac{1}{2} \int_0^{\pi/2} \int_0^{\pi/2} M(\theta, \phi) I(\theta) I(\phi) d\theta d\phi \tag{5.2}$$

where the magnitude of the EMF induced in the element at θ by the current $I(\phi)$ at position ϕ is $\omega M(\theta, \phi) I(\phi) d\theta$. The function $M(\theta, \phi) = M(\phi, \theta)$ is related to L_{dc} by

$$L_{dc} = \int_0^{\pi/2} \int_0^{\pi/2} M(\theta, \phi) d\theta d\phi \tag{5.3}$$

Putting these together with the external capacitance energy storage gives the energy balance equation for the tesla coil:

$$\begin{aligned} &\frac{1}{4} \int_0^{\pi/2} \int_0^{\pi/2} C_{int}(\theta, \phi)(V(\theta) - V(\phi))^2 d\theta d\phi \quad \text{Internal capacitance} \\ &+ \frac{1}{2} \int_0^{\pi/2} C(\theta) V^2(\theta) d\theta \quad \text{External capacitance} \\ &= \frac{1}{2} \int_0^{\pi/2} \int_0^{\pi/2} M(\theta, \phi) I(\theta) I(\phi) d\theta d\phi \quad \text{Inductance} \end{aligned} \tag{5.4}$$

The stored energy and amplitude profiles modeled by the `tsim` program conform to equation 5.4.

The voltage and current profiles in 4.3 for the non-uniform case are also modified by the longitudinal coupling. The following differential equations apply to the amplitude profiles in a tesla coil:

$$\begin{aligned} \frac{dV(\theta)}{d\theta} &= \omega \int_0^{\pi/2} M(\theta, \phi) I(\phi) d\phi \\ \frac{dI(\theta)}{d\theta} &= -\omega C(\theta) V(\theta) - \omega \int_0^{\pi/2} C_{int}(\theta, \phi)(V(\theta) - V(\phi)) d\phi \end{aligned} \tag{5.5}$$

The solution of 5.5 numericaly, using finite differences and gaussian elimination is demonstrated as a function of ω for the lossy case by the `tsim` program. Note that in `tsim` the assumption of phase uniformity is dropped, ie all the V and I are replaced by phasors. The phase uniformity then emerges naturally in the solution to 5.5 at the resonant frequencies.

6. Equivalent parallel capacitance, series inductance

In all cases above, an equivalent parallel capacitance C_{ep} can be assigned to the resonator such that the equivalent capacitance, if charged to V_t would store the same energy as the total energy stored in the resonator's internal and external capacitance. In other words, for the tesla resonator C_{ep} is defined by

(6.1)

$$\frac{1}{2} C_{ep} V_t^2 = \frac{1}{4} \int_0^{\pi/2} \int_0^{\pi/2} C_{int}(\theta, \phi)(V(\theta) - V(\phi))^2 d\theta d\phi + \frac{1}{2} \int_0^{\pi/2} C(\theta) V^2(\theta) d\theta$$

Similarly, an equivalent series inductance L_{es} can be chosen, such that when carrying a current I_b, it stores the same energy as the resonator does when carrying a base current I_b. ie

$$\frac{1}{2} L_{es} I_b^2 = \frac{1}{2} \int_0^{\pi/2} \int_0^{\pi/2} M(\theta, \phi) I(\theta) I(\phi) d\theta d\phi \tag{6.2}$$

Then the equivalents L_{es} and C_{ep} will conform to

$$\begin{aligned} &\frac{1}{2}C_{ep}V_t^2 = \frac{1}{2}L_{es}I_b^2 \\ \text{and therefore} \quad &V_t/I_b = \sqrt{L_{es}/C_{ep}} \end{aligned} \tag{6.3}$$

The equivalents defined in 6.1 and 6.2, which are also valid for toploaded resonators, are functions of both the geometry and the amplitude profiles. If we are prepared to fix the amplitude profiles, we can form simplified expressions for C_{ep} and L_{es} which can be then be related to empirically determined parameters of the resonator.

7. Uniform Current Equivalents - Medhurst

In the 'Medhurst' profile, the current is made approximately uniform by the application of a large endload capacitance, therefore $I(\theta) \approx I_b$ for all θ and, we assume, the voltage profile is the linear rise

$$V(\theta) = \frac{\theta}{\pi/2}V_t \tag{7.1}$$

Under these profile conditions, 6.2 becomes

$$\frac{1}{2}L_{med}I_b^2 = \frac{1}{2}\int_0^{\pi/2}\int_0^{\pi/2} M(\theta,\phi)I_b^2 d\theta d\phi \tag{7.2}$$

which, due to 5.3, simplifies to

$$\frac{1}{2}L_{med}I_b^2 = \frac{1}{2}L_{dc}I_b^2 \tag{7.3}$$

and as expected the equivalent series inductance in the Medhurst case is equal to the DC inductance. Considering now the capacitance, from 6.1, with the linear voltage profile of 7.1, we get for the total energy in the coil capacitance

(7.4)

$$\frac{1}{2}C_{med}V_t^2 = \frac{1}{4}\int_0^{\pi/2}\int_0^{\pi/2} C_{int}(\theta,\phi)\frac{4}{\pi^2}(\theta-\phi)^2V_t^2 d\theta d\phi + \frac{1}{2}\int_0^{\pi/2} C(\theta)\frac{4\theta^2}{\pi^2}V_t^2 d\theta$$

Thus we find the Medhurst equivalent capacitance C_{med} is defined by

$$C_{med} = \frac{2}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2} C_{int}(\theta,\phi)(\theta-\phi)^2 d\theta d\phi + \frac{4}{\pi^2}\int_0^{\pi/2} C(\theta)\theta^2 d\theta \tag{7.5}$$

We can now replace both sides of the energy balance equation 5.4 with Medhurst equivalents to obtain

$$\frac{1}{2}\left(C_{med} + C_{load}\right)V_t^2 = \frac{1}{2}L_{med}I_b^2 = \frac{1}{2}L_{dc}I_b^2 \tag{7.6}$$

where we introduce the endload energy through C_{load} which is a reminder that a substantial endloading is required to produce the Medhurst profile. Since with these amplitude profiles $V_t = \omega L_{dc} I_b$ applies, we have

$$\omega = \frac{1}{\sqrt{L_{dc}\left(C_{med} + C_{load}\right)}} \tag{7.7}$$

The assumption of uniform current is appropriate for most HF inductor applications, and but is of limited value with tesla coils since the endloading condition is poorly met.

8. Cosinusoid Current Profile Equivalents

The uniform current distribution enforced by Medhurst is not available to a tesla coil in normal operation. With no endloading the current profile must fall to zero at the top. We examine the case where the current profile can be approximated by a cosine, $I(\theta) = I_b \cos(\theta)$ and the voltage by the linear rise 7.1. These profiles are reasonable approximations to the actual amplitude distributions of a tesla coil. Disregarding mutual inductance, the stored energy in the inductor is

$$
\begin{aligned}
\frac{1}{2} L_{cos} I_b^2 &= \frac{1}{2} \int_0^{\pi/2} \int_0^{\pi/2} M(\theta, \phi) I_b^2 \cos(\theta) \cos(\phi) d\theta d\phi \\
&\approx \frac{1}{2} \frac{L_{dc}}{\pi/2} \int_0^{\pi/2} I_b^2 \cos^2(\theta) \\
&= \frac{1}{4} L_{dc} I_b^2
\end{aligned}
\tag{8.1}
$$

and so L_{cos}, the lumped equivalent series inductor carrying uniform current I_b storing the same energy as that of the solenoid under the cosine current profile, is given by $L_{dc}/2$. The linear voltage rise allows equation 7.4 to apply and therefore the equivalent capacitance is the Medhurst capacitance 7.5. Thus the energy balance is given by

$$\frac{1}{4} L_{dc} I_b^2 = \frac{1}{2} C_{med} V_t^2 \tag{8.2}$$

To relate V_t to I_b we note that 3.4 applies due to the cosine current profile and the fact that mutual inductance is ignored. Therefore

$$\frac{1}{4} L_{dc} I_b^2 = \frac{1}{2} C_{med} \frac{4}{\pi^2} \omega^2 L_{dc}^2 I_b^2 \tag{8.3}$$

which leads to an approximation for ω:

$$\omega = \frac{\pi}{\sqrt{8}} \frac{1}{\sqrt{L_{dc} C_{med}}} \tag{8.4}$$

These profiles are fair approximations for a tesla coil with little or no topload. When heavy loading is present, the current distribution lies somewhere between the cosine and the uniform models and therefore the resonant frequency is expected to lie between the extremes given by 7.7 and 8.4.

9. Forward transimpedance

The ratio of top voltage to base current is an important metric of a tesla coil and is of particular relevance to CW coils and can be related to the equivalent reactances. For the lumped LC model, the forward transimpedance is

$$Z_{ft} = \frac{V_t}{I_b} = \omega L \tag{9.1}$$

and the above equation also applies to a distributed line with L replaced by L_{dc} if there is sufficient end loading to make the current almost uniform.

For a tesla coil with a cosine current profile and a linear voltage profile we can, from 8.2 and 3.4, write

$$Z_{ft} = \frac{V_t}{I_b} \approx \sqrt{\frac{1}{2} \frac{L_{dc}}{C_{med}}} \approx \frac{2}{\pi} \omega L_{dc} \tag{9.2}$$