A PRECISION MODEL OF A CLOSEWOUND SOLENOID.

PAUL NICHOLSON

Version 0.2, see [17] for the latest version of this document and the accompanying software.

Warning This document not been independently checked or peer reviewed by any authoritative body, so please exercise a due level of scepticism. The author welcomes criticism.

1. Introduction

This article describes a detailed software model of a closewound solenoid operating perpendicular to a ground plane - the configuration typicaly used as a resonating secondary in tesla coils [11].

The model is based on a non-uniform transmission line implemented as a large network of small discrete LCR sections. We extend the usual transmission line equivalent circuit element to take account of the extra longitudinal couplings present in a solenoid. Additional elementary current sources are introduced in shunt to represent the displacement currents due to internal capacitance, and additional voltage sources are inserted in series to represent EMF induced through mutual coupling between turns. The mutual inductance matrix is obtained by summing circular current filaments and the internal and external capacitance matrix is built up by repeated high resolution solutions of the laplace equation for the surrounding field.

The resulting network is exactly solved in the frequency domain for the AC steady state conditions to produce the input impedance spectrum and amplitude profiles at each frequency. The model contains only two arbitrary constants which specify loss factors, and requires input measurements of only lengths, turns, and one DC resistance measurement.

A reference implementation in C is provided which is able to accurately predict the location of the odd-wave resonances up to the 5/4 wave overtone [1]. Estimation of Q factor and input impedance is poor due to the use of a naive approximation to the effective AC resistance of the solenoid.

Date: 23rd August 2000.

2. Summary of terms and definitions

The term *section* refers to the elementary LCR sections of the transmission line. The term *segment* refers to regions of the coil distinct in the capacitance matrix.

(2.1)

N_s	Number of elementary LCR sections.
N_t	Number of turns on the solonoid, rounded to nearest integer.
N_{cp}	Number of segments used in the capacitance matrices.
S	Number of LCR sections per turn of the coil, currently one.
R_s	The AC series resistance of each LCR section, uniform.
$C_{ext}(k)$	External capacitance of coil segment k.
$C_{int}(j,k)$	Internal capacitance between coil segments j,k.
$C_{tor}(j)$	Internal capacitance between coil segment j and topload.
$C_s(u,v)$	Internal capacitance between LCR sections u,v.
$I_s(u)$	Total displacement current into section u due to C_s.
$Y(k)$	Shunt admittance of the kth LCR section.
$M_{pt}(u,v)$	Mutual inductance between sections u and v.
L_{coil}	DC inductance of the coil.
W	Winding loss multiplier - fiddle factor one.

3. Series resistance

The resistance of a wire conductor increases at high frequencies due to confinement of current to a surface layer. Formula such as

$$R_{ac} = R_{dc}\left(1 + \sqrt{\frac{2\pi\mu f r_{wire}^2}{17.3 \times 10^{-9}}}\right) \tag{3.1}$$

are derived for use with straight conductors [1]. In a close wound coil, the magnetic field causes the surface currents to migrate towards the inner surface of the winding and thus the AC resistance is increased beyond that expected by 3.1. Medhurst [4] provides a table of empirical factors ϕ by which the straight conductor AC resistance must be multiplied when the conductor is wound into a solenoid. However the ϕ factor is only valid below the lowest resonant frequency. As a temporary measure, we accept the use of ϕ but introduce an additional arbitrary winding loss factor W. We compute the straight line AC resistance of each line section using 3.1 and multiply by the Medhurst factor ϕ interpolated from table VIII in [4], and further multiply by W to obtain an estimate of the uniform effective AC series resistance per section R_s.

4. Inductance Profile

The self inductance L_n which a single turn would have if measured in isolation is approximated by the formula [1],

$$L_n = \mu_{air} r_{coil}\left(\ln\left(\frac{8r_{coil}}{r_{wire}}\right) - 2\right) + \frac{\mu_{copper} r_{coil}}{4} \tag{4.1}$$

where r_{coil} and r_{wire} are radii in metres. When embeded in a solenoid the EMF induced in this turn will be augmented by the presence of flux from neighboring turns - an effect which will be smaller nearer the ends of the coil since there are fewer neighbouring turns. We can describe the EMF induced in the kth turn of a N turn coil by

$$e_k = \sum_{r=1}^{N} M_{pt}(k,r)\frac{di_r}{dt} \tag{4.2}$$

where i_r is the current in the rth turn and the matrix $M_{pt}(k,r)$ of mutual inductances is introduced to characterise the distribution of flux linkage along the coil. M_{pt} has the following properties

$$M_{pt}(r,s) = M_{pt}(s,r) \tag{4.3}$$

$$M_{pt}(r,r) = L_n \tag{4.4}$$

where we assume a uniform self inductance L_n per turn as given by 4.1. For a long narrow coil we might expect non-neglible values of M_{pt} to be concentrated in a narrow band along the diagonal of the matrix, and as the coil length/width ratio is reduced this band would presumably widen. We can estimate the shape of M_{pt} by applying a standard formula [1] for the mutual inductance between coaxial circular current filaments, so that between sections n and m we have mutual inductance.

$$M_{pt}(n,m) = \sigma\frac{\mu}{4\pi}r_{coil}F \quad \text{if } n \neq m \tag{4.5}$$

where σ is a normalising factor described below, and F is a factor obtained from tables and is a function of the coil radius and filament separation.

We would now be required to integrate 4.5 over the penetrated area of the conductor but this can be neatly avoided by recognising that, if we assume the conducting area is uniform along the wire, then 4.5 is correct apart from a normalising factor which we apply through σ. M_{pt} can be normalised by noting that at low enough frequencies the currents in each turn can be considered the same, so the total induced voltage V_{coil} is, from 4.2,

$$V_{coil} = \sum_{j=1}^{N} e_j = \frac{dI_{coil}}{dt}\sum_{j,k=1}^{N} M_{pt}(j,k) \tag{4.6}$$

where I_{coil} is the uniform current. Thus the total inductance L_{coil} at low frequency is related to M_{pt} by

$$L_{coil} = \sum_{j,k=1}^{N} M_{pt}(j,k) \tag{4.7}$$

The Lundin formula [14] for the Nagaoka inductance provides L_{coil} and with an estimate for L_n from 4.1 we can choose a value of σ such that 4.7 is satisfied. By this straightforward means we construct a good estimate of M_{pt} for use in a non-uniform transmission line model.

Since in our model the line is represented by S sections per turn, we can replace the series inductance L in the kth LCR section with a voltage source given by,

$$V_m(k) = \frac{jw}{S}\sum_{r=1}^{N_s} M_{pt}(1+\lfloor\frac{k-1}{S}\rfloor, 1+\lfloor\frac{r-1}{S}\rfloor)I(r) \tag{4.8}$$

in which the index expressions translate from section numbers to turn numbers.

5. External Capacitance

We define *external* capacitance as that resulting from flux paths from any part of the coil or topload to ground. The remainder is defined as *internal* capacitance and includes flux paths between regions of the coil and itself, and between regions of the coil and topload. This distinction is convenient rather than fundamental, and is reflected in the way the two capacitances are implemented. The capacitance matrix for the resonator, topload and surroundings, is obtained from a laplace relaxation solver in a separate program [5]. The laplace solver exploits the axial symmetry and employs 3D cylindrical coordinates to carry out a pseudo-2D solution. By partitioning the winding into a stack of short independent cylindrical conducting electrodes we can sum the flux arriving at each of these coil segments separately, which then allows the capacitance distribution of a particular coil to be determined in terms of the equivalent lumped capacitance that each segment of the coil has to ground.

The remainder of this section describes external capacitance - internal capacitance will be dealt with in the next section.

The laplace solver provides the external capacitance as an array C_{ext} of size N_{cp}, each element being the total capacitance to ground from the corresponding segment of the coil. The array is indexed $1..N_{cp}$ working from the base of the coil upwards [6].

Then, since the coil is simulated in N_s sections, the shunt admittance to ground from the kth LCR section is

$$Y(k) = G(k) + jwC_{ext}(1 + \lfloor N_{cp}\frac{k-1}{N_s}\rfloor) \quad ; \quad k = 1\ldots N_s \tag{5.1}$$

in which the index expression maps LCR section numbers to coil segment numbers. We would like the shunt conductance $G(k)$ to account for energy absorbed into the surroundings from the electric field, but without a means to determine this we will simply assign a constant dissipation factor D_p so that

$$Y(k) = \frac{N_{cp}}{N_s}(D_p + j)wC_{ext}(1 + \lfloor N_{cp}\frac{k-1}{N_s}\rfloor) \quad ; \quad k = 1\ldots N_s \tag{5.2}$$

This equation will be used in section 7.

6. Internal Capacitance

Palermo [9] described the capacitance between adjacent turns by using the formula for capacitance between two parallel conductors, supported by some rather dubious experimental results. In this model, the total capacitance between adjacent turns is

$$\begin{aligned} C &= \frac{2\pi^2\epsilon r_{coil}}{\ln a + \sqrt{a^2-1}} \\ a &= \frac{1}{\text{spacing factor}} = \frac{\text{winding pitch}}{\text{conductor diameter}} \end{aligned} \tag{6.1}$$

Subsequent measurements by Medhurst [4] and others have not been able to reproduce anything like the expected internal capacitance and this has been attributed to migration of current flow to the inner surface of the winding, thus drawing away from the inter-turn gap [15]. The 'Palermo' capacitance is included in the model as an option in order to examine its effect. The capacitance between more widely space turns is obtained from the laplace solver mentioned in the previous section. With the coil partitioned into N_{cp} segments we obtain a symmetric array $C_{int}[1\dots N_{cp}][1\dots N_{cp}]$ such that $C_{int}[a][b]$ is the total equivalent lumped capacitance between coil segments a and b. This approximation is likely to break down for adjacent sections, so these are not used. As a result the internal capacitance of the coil is likely to be slightly under-represented in the model. The model introduces additional currents into each section to represent the movement of charges induced through the internal flux. If the equivalent capacitance between any two transmission line sections u, v can be regarded as a function C_s of u, v then the section u has a current $I_s(u)$ injected from the rest of the winding due to C_s given by

$$I_s(u) = jw \sum_{r=1}^{N_s} C_s(u,r)(V(r) - V(u)) \tag{6.2}$$

We can construct an estimate of C_s from the following approximations [8]:

$$\begin{aligned}
r &= 1 + \lfloor N_{cp} \frac{u-1}{N_s} \rfloor \quad , \quad s = 1 + \lfloor N_{cp} \frac{v-1}{N_s} \rfloor \\
C_s(u,v) &= 0 \quad \text{if } |u-v| < S \\
C_s(u,v) &= \frac{1}{S} \frac{2\pi^2 \epsilon r_{coil}}{\ln\left(a + \sqrt{a^2-1}\right)} \quad \text{if } |u-v| = S \text{ and Palermo enabled} \\
&\quad \text{where } a = 1/\text{spacing factor} \\
C_s(u,v) &= 0 \quad \text{if } r = s \\
C_s(u,v) &= \frac{N_{cp}^2}{N_s^2} C_{int}(r,s) \quad \text{if } r \neq s
\end{aligned} \tag{6.3}$$

If $r = s$ then the line sections u, v are in the same coil segment and the C_{int} array gives us no information.[7]

The top loading toroid also exhibits considerable distributed internal capacitance to the coil which can be treated in a similar way by injecting a current $I_t(k)$ into the kth LCR section. The toroid voltage is $V(N_s+1)$ and we assume the internal toroid capacitance distribution is given by an array $C_{tor}(r)$ where r is the coil segment number. Then

$$\begin{gathered}
I_t(k) = jw \frac{N_{cp}}{N_s} C_{tor}(r)(V(N_s+1) - V(k)) \\
\text{where } r = 1 + \lfloor N_{cp} \frac{k-1}{N_s} \rfloor
\end{gathered} \tag{6.4}$$

7. The extended transmission line model

Our starting point is the conventional transmission line model built from a cascade of small discrete LCR sections, as shown in figure 1. There are S sections per turn, so each section has length $1/S$ turns, and for later convenience we require $S \geq 1$ to be an integer [2]. There are therefore $N_s = N_t S$ sections in a coil of N_t turns, and these are labelled $1 \dots N_s$.

The series inductance is replaced by a voltage source $V_m(k)$. Current injected due to internal capacitance from the rest of the winding is represented by a current source $I_s(k)$, and that due to the toroid by a current source $I_t(k)$.

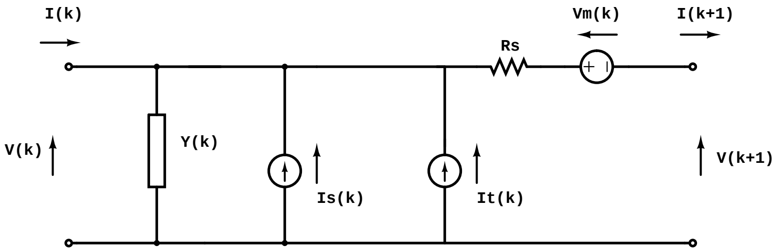

Fig 1. Transmission line section with additional sources.

The steady state AC voltages and currents for the kth section are

$$I(k+1) = I(k) - Y(k)V(k) + I_s(k) + I_t(k) \quad ; \quad k = 1 \dots N_s \tag{7.1}$$

$$V(k+1) = V(k) - RI(k+1) - V_m(k) \quad ; \quad k = 1 \dots N_s \tag{7.2}$$

in which $V_m(k)$ is given by 4.8; $I_s(k)$ is given by 6.2; $I_t(k)$ is given by 6.4; and the shunt admittance $Y(k)$ is given by 5.2. A termination Z_l containing the lumped external toroid capacitance follows the final section N_s, giving

$$V(N_s+1) = Z_l \left(I(N_s+1) - \sum_{k=1}^{N_s} I_t(k) \right) \tag{7.3}$$

where we deduct the displacement currents which service the toroid to coil capacitance. Finaly, a reference voltage into the first section provides

$$V(1) = V_{in} \tag{7.4}$$

Substituting 7.4 into 7.2 and eliminating the recursion gives

$$V(k) = V_{in} - R\sum_{r=2}^{k} I(r) - \sum_{r=1}^{k-1} V_m(r) \quad ; \quad k = 2 \dots N_s + 1 \tag{7.5}$$

and

$$V(a) - V(b) = -R \sum_{r=b+1}^{a} I(r) - \sum_{r=b}^{a-1} V_m(r) \quad ; \quad a > b \tag{7.6}$$

Substituting 7.3, 7.5, and 7.6 into 7.1 eliminates all the voltages and leaves a set of $N_s + 1$ simultaneous equations in the $N_s + 1$ currents $I(k)$

$$[A_{jk}]\,[I_k] + [C_j] = [0] \quad ; \quad j, k = 1 \dots N_s + 1 \tag{7.7}$$

The substitutions leading to 7.7 involve a lot of coefficients and are conveniently done by the software during the construction of $[A_{jk}]$. The system 7.7 can then be solved by gaussian elimination to obtain the current distribution $[I_k]$ and the voltage distribution via 7.5.

The rate of energy loss is then available through

$$\begin{aligned} P_c &= \sum_{k=1}^{N_s} |V(k)|^2 \Re(Y(k)) \quad \text{Power loss due to Dp} \\ P_w &= \sum_{k=1}^{N_s} |I(k+1)|^2 Rs \quad \text{Copper loss in winding} \\ P_l &= |I(N_s+1)|^2 \Re(Zl) \quad \text{Power absorbed in the load} \end{aligned} \tag{7.8}$$

and as a cross check on the computation, we expect the input power

$$P_{in} = |I(1)|^2 * \Re(Zin) = P_c + P_w + P_l \quad (7.9)$$

The program carries out the above substitutions and elimination for each required frequency point to generate the output stream described in section 9.

8. Installation and Operation

Begin by unpacking the distribution file with, say,

```
tar xpzf tssp-v0.2.tgz
```

and compile the software according to the contents of the `README` file. Take a look at the example resonator description files `*.in` and copy one to a new file, say `arthur.in` and edit to suit your resonator. Next, run the laplace solver on the system,

```
./tlap arthur
```

This program will generate capacitance matrix data into a plain text file `arthur.caps`. You can also watch what its doing on the X display - the yellow areas are regions where the field is still unsettled. Once `tlap` has finished [16] you have a completed `arthur.caps` in the working directory, which is ready for use by the modelling program. This has the syntax

```
./tsim yoursys [from Hz] [to Hz] [step Hz]
```

which will look for a description file `yoursys.in` and a capacitance file from tlap called `yoursys.caps`. The remaining arguments in integer Hz specify the frequency range and step size, and the results appear on `stdout`. For example,

```
./tsim arthur 35000 420000 5000 > arthur.data
```

which will run the model on the input file `arthur.in` over a frequency range 35kHz to 420kHz in 5kHz steps. This may take some time to run, so progress can be checked from time to time with a command like

```
grep FREQ < arthur.data | sort -nu +1.0
```

If several processors are available, more than one copy of `tsim` may be run simultaneously, eg,

```
./tsim arthur 50000 400000 4000 > /tmp/t1 &
./tsim arthur 51000 400000 4000 > /tmp/t2 &
./tsim arthur 52000 400000 4000 > /tmp/t3 &
./tsim arthur 53000 400000 4000 > /tmp/t4 &
```

which interleaves the work to produce 1kHz steps. The results can be merged at any time with

```
cat /tmp/t[1-4] > arthur.data
```

Once a program run has completed and the results have been examined, additional data can be appended to an existing output file by further runs of `tsim` in order to generate additional resolution around points of interest. The results can be viewed at any time by examining the output file, the format of which is described in section 9. A filter utility `zplot` is provided with the package which will produce a simple set of impedance charts in postscript when an output file is piped into it, eg,

```
./zplot < arthur.out > arthur-z.ps
gv arthur-z.ps
```

The `zplot` program may be used while `tsim` is still running and it will use whatever data has been generated so far. A similar utility generates voltage and current distribution charts for a given frequency,

```
./viplot 40000 < arthur.out > arthur-vi40k.ps
gv arthur-vi40k.ps
```

assuming `arthur.out` contains records for 40kHz.

An MPI version of the software is available for use on Beowulf clusters.

9. Output format

Program `tsim` writes all its results to the `stdout` stream which should be redirected to a file. The output file format is plain text, newline separated records, space separated fields with multiple record types identified by keyword in the first field. Record types are as follows:

Frequency domain records: One of these is issued for each frequency point examined. These have the following fields:

1. The keyword FREQ.
2. Frequency, Hz.
3. $\Re(Z_{in})$, the real part of the input impedance.
4. $\Im(Z_{in})$, the imaginary part of the input impedance.
5. $|V(N_s+1)|/|V(1)|$, the magnitude of the voltage gain.
6. $100 \times P_c/P_{in}$, percentage power dissipated in external electric field.
7. $100 \times P_w/P_{in}$, percentage power dissipated in winding.
8. $100 \times P_l/P_{in}$, percentage power dissipated in terminating load.
9. The Q factor.
10. P_{err}.

where P_{err} is an end-to-end check based on 7.9,

$$P_{err} = 100.0 \times \frac{P_l + P_c + P_w - P_{in}}{P_{in}} \tag{9.1}$$

which may be taken as an estimate of the percentage error of the substitution and elimination arithmetic.

VI distribution records: As each frequency point is visited, the program writes out the complete voltage and current distribution as N_s+1 records of the following format.

1. The keyword VI.
2. Frequency, Hz.
3. k, the LCR section number $1\ldots N_s+1$.
4. $Re(V(k))$, the real component of the voltage on section k.
5. $Im(V(k))$, the imaginary component of the voltage on section k.
6. $Re(I(k))$, the real component of the current on section k.
7. $Im(I(k))$, the imaginary component of the current on section k.

Winding resistance records: One of these is issued for each frequency point examined, to provide a record of the AC resistance values chosen by the program. These have the following fields:

1. The keyword RAC.
2. Frequency, Hz.
3. R_{ac}, total AC straight conductor resistance of winding.
4. $R_{ac} \times \phi$, total AC wound resistance before the W factor.

	$f1$	$f3$	$f5$	$f3/f1$	$f5/f1$
Frequencies - Test Coil	66.7kHz	193.3kHz	307.0kHz	2.90	4.60
Frequencies - Model	66.5kHz	194.6kHz	306.5kHz	2.93	4.61
Z_{in} Test Coil	120 ohms	470 ohms	905 ohms		
Z_{in} Model	68 ohms	105 ohms	130 ohms		

TABLE 1. Comparison of test coil with model

Data can be extracted easily from this results file which is intended for easy post-processing with standard utilities. For example, if the frequency domain information is required, the following shell command extracts the data:

```
grep FREQ < arthur.data | sort -nu +1.0 > /tmp/freqdata
```

from which the impedance magnitude can be plotted with the gnuplot command

```
plot "/tmp/freqdata" using ($2):(sqrt($3*$3+$4*$4))
```

The scripts `zplot` and `viplot` can also be inspected for further examples.

10. Results

It is useful to describe the response of the resonator in terms of the input impedance at the base of the coil, as the location and depth of the minima of this response can be reliably compared with actual coils [3]. Table 1 shows a typical comparison with a test coil [17]. The model provides a reasonably accurate prediction of the resonant frequencies and has been successful with all coils tested.

If the inter-winding capacitance is deliberately over-applied in the model, excessive dispersion results. If mutual inductance is under-applied, excessive dispersion results. The effect of these two perturbations is diminished when a toroid is fitted. The results shown above are generated without the 'Palermo' inter-turn capacitance. If this is enabled, the predicted frequencies drop by around 1% at the 1/4 wave, and roughly proportionaly more at the higher resonances. This effect is also diminished on coils fitted with toroids. The results are slightly better overall with the Palermo capacitance enabled, but the difference is within experimental limits and thus we cannot draw any firm conclusion from this.

The results so far presented suggest that the energy storage mechanisms of solenoids as currently understood do bear up to numerical scrutiny. The solenoid, if modelled as a transmission line in sufficient detail, gives an accurate prediction of resonant frequencies up to 5/4 wave resonance. There is no reason to suppose that higher order resonances cannot also be predicted. The inability to predict input impedance and Q factor without the use of arbitrary coefficients is a reminder that a theory of resonator winding loss is still lacking.

11. Notes

[1] "Electronic Designers Handbook", Giacoletto, 1977.

[2] Currently the model uses S=1, ie one LCR section per turn.

[3] We ignore the poles as their position is highly sensitive to stray reactive loading at the base due to connection to test equipment.

[4] "HF Resistance and Self-Capacitance of Single-Layer Solenoids", R.G. Medhurst, Wireless Engineer, Feb and March 1947.

[5] A laplace solver program is provided in the distribution which will generate the input arrays required for the simulator.

[6] Currently N_{cp} is set to 32.

[7] Equ 6.3 as it stands slightly over-emphasises the inter-winding capacitance. The program retains the spirit of 6.3 but uses some tricks to provide interpolation.

[8] This only makes sense for S=1 and possibly S=2.

[9] A.J. Palermo, 'Distributed Capacity of Single-layer Coils', Proc IRE 1934, Vol 22, p897.

[11] `http://www.webring.org/cgi-bin/webring?ring=TeslaRing&list`

[14] R. Lundin, 'A Handbook formula for the Inductance of a Single-Layer Circular Coil', Proc IEEE, Vol 73, 1428-1429.

[15] I'm really not convinced by this traditional argument: so what if the current flow migrates inwards, surely the turn-to-turn PD is still impressed across the gap?

[16] May take a few days to run. Has to do at least 32 complete relaxations of the field - a different electrode energised each time.

[17] Refer to http://www.abelian.demon.co.uk/tssp/ for the latest version of this document, for further test results, and for the source code.